Local Stability and Bifurcations in Kaldor Model[#]

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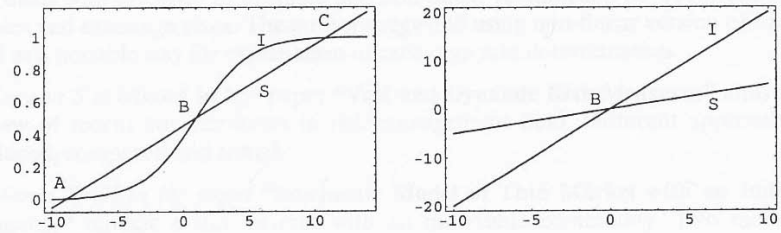
1. Introduction to Kaldor model

Nicholas Kaldor (1940) introduced one of the most interesting theories of business cycle. It is distinguishable from most other contemporary treatments since it utilizes non-linear functions, which produce endogenous cycles. Until Kaldor (1940) the general economic treatment was rather linear. The problem is, that linear relations between economic variables always lead to one steady state as two lines can never intersect in two points. This was obviously incompatible with the empirical reality of cycles and fluctuations. To model cycles more than one steady state is needed. Kaldor (1940) therefore introduced nonlinear relations. In particular he assumed that investment and saving functions will nonlinearly depend on income and capital and . Fig. 1 shows the difference between linear and nonlinear treatment.

Fig. 1: Comparison of Nonlinear and Linear Relations



With nonlinearity assumption on investment there can be up to three fixed points {A, *B, C*}.[1] However, whether there will be one or three fixed (equilibrium) points depends on the slope of both functions. So even under assumption of nonlinear investment and savings if the slope of saving function is too high there still is only one equilibrium point. Same conclusion also holds for case with linear savings. Fig. 2 provides the intuition.

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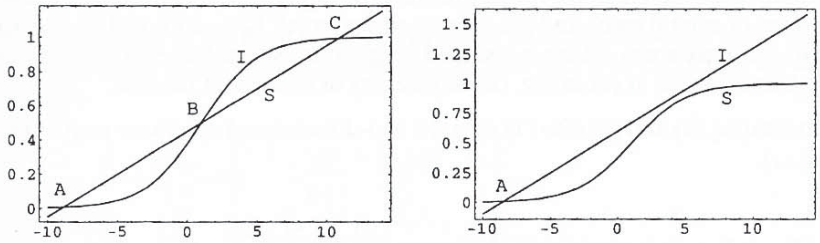
1 These points represent economic equilibrium, i.e. points where investment equal savings.

Fig. 2: One vs. three equilibrium points in case of nonlinear investment and linear savings



Thinking of the dynamic behavior when investment is above saving function left of A, income increases towards A. Right of A where investment is below saving function, income decreases towards A and vice versa. Consequently we can see that fixed point B is unstable (income never converges to it) while points A and C are stable equilibria (income always converge to either of them). It results that nonlinear version with three equilibrium points leads to bistability i.e. cyclical behavior. Kaldor argued that oscillating behavior in economy arises when we assume nonlinear investment. But this is not the only condition to be fulfilled to reach cyclical motion. So called *Speed of Adjustment* must be introduced to have complete set conditions for oscillation. The speed of adjustment represents firms' reaction to excess demand. If the reaction to an increased demand is slow and firms are not capable to offset the increase it may be likely that economy will never leave one of the equilibriums. On the other hand if the reaction is rapid economy can be constantly switching between the two equilibrium points {A, *C*}.

2. The Model

Let us consider the following discrete-time version of the Kaldor model,

$$\begin{aligned} Y_{t+1} &= Y_t + \alpha(I_t - S_t), \\ K_{t+1} &= (1-\delta)K_t + I_t, \end{aligned} \tag{1}$$

where Y_t and K_t denotes the output level and the capital stock in period t. As mentioned above an important parameter in the Kaldor model is the speed of adjustment α which represents firm's reactions to the demand excess. Parameter α is restricted to be greater than zero. A small value of α means slow firms' reaction, which can be explained by a high degree of risk aversion or a relevant monopoly degree (Bischi, 2001). In contrast, a high value of α means fast reactions. The parameter δ ($0 < \delta < 1$) measures capital stock's depreciation rate. Savings are proportional to the level of income,

$$S_t = \sigma Y_t, \tag{2}$$

where σ ($0 < \sigma < 1$) denotes the propensity to save. Nonlinearity of the model arises from the sigmoid shaped investment function,

$$I_t = \sigma\mu + \left(\frac{1}{1+\exp\left[(-\lambda(Y_t - \mu)\right]} - \frac{1}{2} \right) + \frac{\gamma}{1+r}\left(\frac{\sigma\mu}{\delta} - K_t \right), \tag{3}$$

where the parameter μ is the normal level of income exogenously assumed in firms expectations, $\sigma\mu/\delta$ then denotes the normal level of capital stock. Adjustment between the normal level of capital stock and the current capital stock K_t is controlled by a coefficient γ divided by the interest rate, which is assumed exogenous. Normal level of savings is defined as $\sigma\mu$. λ defines degree of concavity, i.e. nonlinearity of investment function.

Substituting (2) and (3) into (1) we get a two-dimensional nonlinear map T: $(Y_t, K_t) \rightarrow (Y_{t+1}, K_{t+1})$,

$$T:\begin{cases} Y_{t+1} = Y_t + \alpha\left[\sigma\mu + \left(\dfrac{1}{1+\exp\left[(-\lambda(Y_t-\mu)\right]} - \dfrac{1}{2}\right) + \dfrac{\gamma}{1+r}\left(\dfrac{\sigma\mu}{\delta} - K_t\right) - \sigma Y_t\right] \\ K_{t+1} = (1-\delta)K_t + \sigma\mu + \left(\dfrac{1}{1+\exp\left[(-\lambda(Y_t-\mu)\right]} - \dfrac{1}{2}\right) + \dfrac{\gamma}{1+r}\left(\dfrac{\sigma\mu}{\delta} - K_t\right) \end{cases} \quad (4)$$

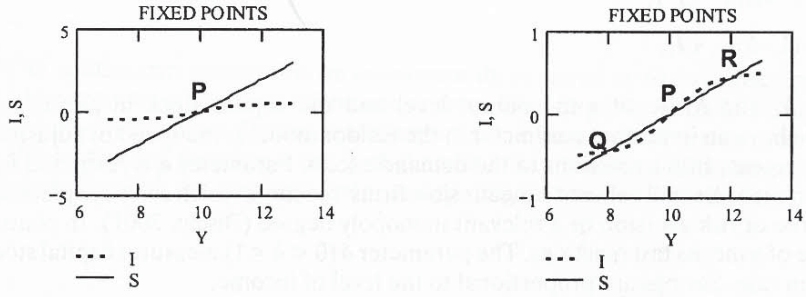
For obtaining the equilibrium points we set $Y_{t+1}=Y_t$ and $K_{t+1}= K_t$. Then we have:

$$K = \frac{\sigma}{\delta} Y,$$

$$\sigma\mu + \left(\frac{1}{1+\exp\left[(-\lambda(Y_t-\mu)\right]} - \frac{1}{2}\right) = \frac{1}{1+r}\frac{\gamma\mu}{\delta}(Y-\mu). \quad (5)$$

From equations (5) it is clear that we can have one or three fixed points. The unique steady state P = $(\mu, \mu\sigma/\delta)$, or P and two further steady states R = $(Y_R, \sigma/\delta Y_R)$ Q = $(Y_Q, \sigma/\delta Y_Q)$, located symmetrically to the steady state P (see Fig. 3). The values Y_R and Y_Q can be computed from Eq. (5) as the smallest real solution of the second equation.

Fig. 3: Left: One fixed point P, σ = 0.257 Right: Three fixed points PQR, σ = 0.057



2.1 Local stability

The local stability of the fixed point P = (μ, $\mu\sigma/\delta$) is obtained through the localization on the complex plane of the eigenvalues of the Jacobian matrix of the map T for the fixed point P, Eq (4).

$$J\left(\mu, \mu\frac{\sigma}{\delta}\right)=\begin{pmatrix} 1+\alpha\left(\frac{\lambda}{4}-\sigma\right) & -\frac{\alpha\gamma}{1+r} \\ \frac{\lambda}{4} & 1-\frac{\gamma}{1+r}-\delta \end{pmatrix}. \tag{6}$$

The eigenvalues of (6) are the solutions of the characteristic equation:

$$P(x)=x^2-tr(J)x+Det(J)=0. \tag{7}$$

The necessary and sufficient conditions for the two roots of Eq. (7) to be inside the unit circle of the complex plane are expressed by the system of inequalities:

$$\begin{aligned} &P(1)=1-Tr(J)+Det(J)>0, \\ &P(-1)=1+Tr(J)+Det(J)>0, \\ &P(0)=Det(J)<1. \end{aligned} \tag{8}$$

The first condition can be formulated as

$$\sigma>\frac{(r+1)\delta\lambda}{4(\gamma+r\delta+\delta)}, \tag{9}$$

then there is the unique fixed point P = (μ, $\mu\sigma/\delta$), if this condition does not hold then map T has three fixed points and the central fixed point P is not stable. The second condition (8) becomes

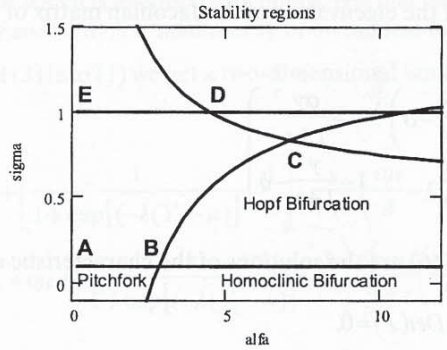
$$\sigma<\frac{8\gamma+r(\delta-2)(\alpha\lambda+8)+(\delta-2)(\alpha\lambda+8)}{4\alpha(\gamma+r(\delta-2)+\delta-2)}. \tag{10}$$

The last condition of (8) can be expressed as

$$\sigma>\frac{\frac{\gamma}{r+1}+\delta+\frac{1}{4}\alpha(\delta-1)\lambda}{\alpha\left(\frac{\gamma}{r+1}+\delta-1\right)}. \tag{11}$$

Inequalities (9), (10), (11) define the stability region for the fixed point P. The region ABCDE in Fig. 4 represents the stability area.

Fig. 4: Stability region ABCDE of the steady state P in the plane of parameters. This figure is obtained using parameters $\gamma = 0.6$, $\delta = 0.2$, $\lambda = 2.5$ and $r = 0.05$.



Below the AB line, three equilibria exist and the situation of bi-stability arises, this is a typical pitchfork bifurcation, Lorenz (1993). On the boundary line BC, we have Det(J) = 1, i.e. two complex roots with modulus equal to one. If the point (α, σ) in Fig. 4 crosses the boundary BC line a Hopf bifurcation occurs, Lorenz (1993).

3. Dynamic Analysis

Now that we have defined the model we proceed with analysis of dynamic properties. To carry out the analysis we set the parameters to following values: $\gamma = 0.6, \delta = 0.2, \lambda = 2.5$ and $r = 0.05$. α and σ during the analysis. Their particular values will be always given for each of the cases.

3.1. Dynamic behavior in one equilibrium case

In this section we examine dynamic behavior in one equilibrium setting. To obtain one equilibrium framework the slope of the saving function must be $\sigma > 0.162$.[2] We set it to $\sigma = 0.25$, which is surely above the critical level. We can now investigate the impact of changes in *speed of adjustment* - α. Later we will explore whether the model settled in the equilibrium point B or not.

Let's begin with the trivial case of low α, i.e. firms' reaction to demand increases is slow. We set $\alpha = 0.35$ which is a point left of the BC curve[3]. As we see from the Fig. 5 resulting combinations of $\{Y, K\}$ create tim paths[4] converging to equilibrium point B. There is clearly no cyclical behavior which means that system eigenvalues are real numbers ($\lambda_1 = 0.9604$ and $\lambda_2 = 0.3993$). Intuition behind this is simple. As firms react slowly income and capital evolve gradually and overshooting is absent. The different colors in the plot

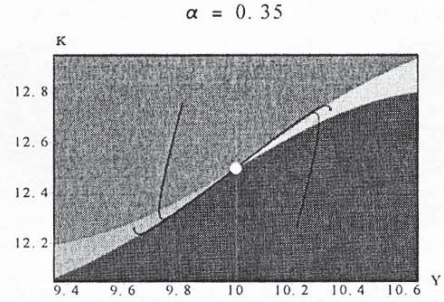
2 In other words it must be in the stability region as described in the previous section.

3 Again, well inside the stability region.

4 Time path is a solution of system of two differential equations given in (1.5). They show us where the system will end up if we choose particular initial values for Y and K.

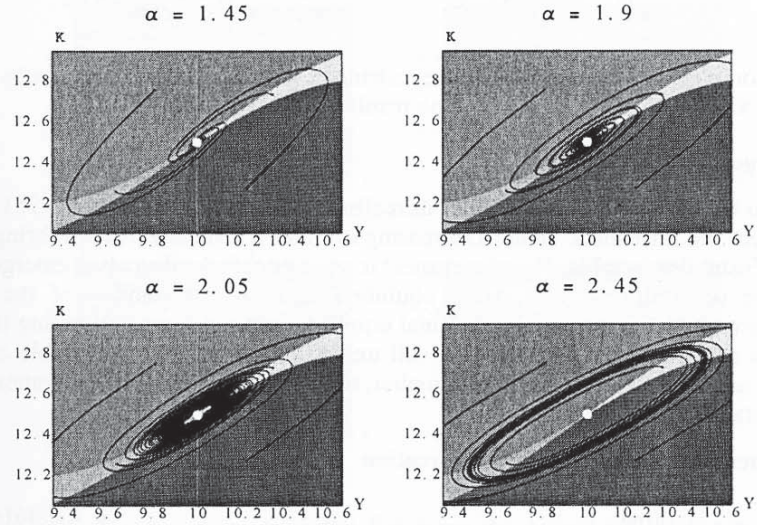
area separate areas of different time paths directions.[5] These are obviously areas of four different system motion directions.

Fig. 5: Trivial example of one equilibrium state



As a next step we gradually increase speed of adjustment and observe consequent changes in the system dynamics. α is set to 1.45, 1.9, 2.05 and 2.45 respectively. It is expected that the $\{Y, K\}$ paths will exhibit cyclical behavior as α increases. See Fig. 6 for the results.

Fig. 6: Y and K time paths under varying speed of adjustment

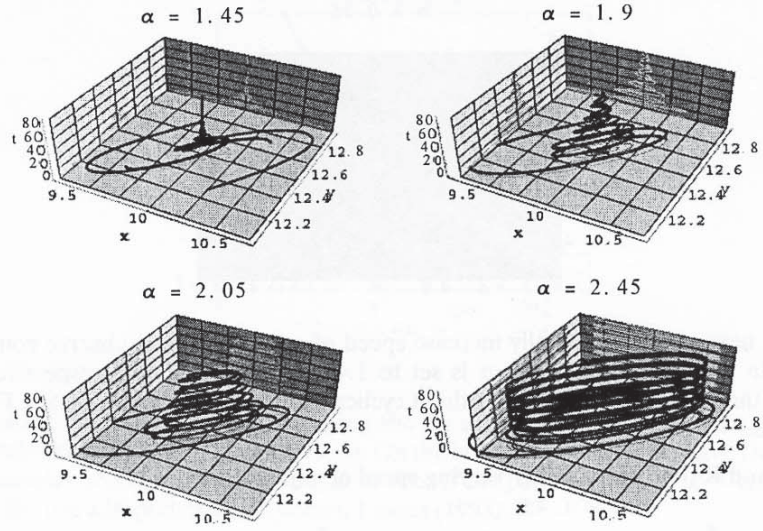


The above pictures deliver a clear message. The higher the speed of adjustment the stronger overshooting and consequent cyclical pattern. For all α given above the paths cycle around the equilibrium point B. That means that eigenvalues are complex conjugate. In particular for $\alpha = 1.45$ the eigenvalues are: $\lambda_1 = 0.886161 + 0.29229i$ and $\lambda_2 = 0.886161 - 0.29229i$. Obviously, as $\lambda_{1,2} > 1$ the system converges. Please note the last two pictures in the series ($\alpha = 2.05$ and $\alpha = 2.45$). We witness here an emergence of the Hopf bifurcation. We see that the system no longer converges to equilibrium B but to a closed orbit instead. With $\alpha = 2.45$ this dynamic pattern is even stronger and the orbit is

5 The separation lines are called null-clines. These are lines where neither Y nor K change in time, i.e. $\dot{Y}=0$, resp. $\dot{K}=0$.

observed clearly. In this case, whatever the initial stock of capital and income the economy ends up cycling around the equilibrium point.

Fig. 7: Time paths in one equilibrium setup with increasing speed of adjustment.

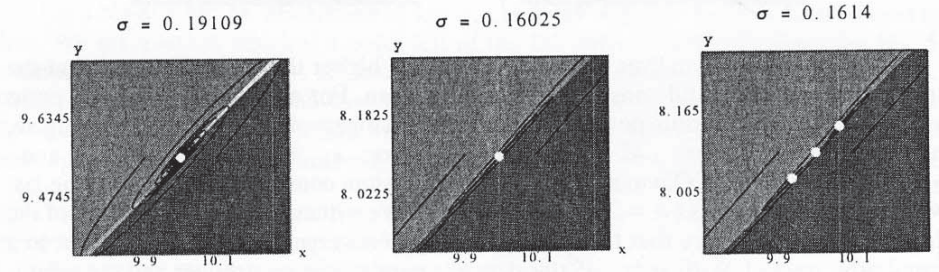


To complete this picture we have calculated corresponding time paths for the particular values of α. Fig. 7 presents the results.

3.2. Emergence of two equilibria

So far we have worked with rather unrealistic propensity to save ($\sigma = 0.25$). We will now correct the unrealistic value. Decreasing σ under a critical value will bring us to a region of instability (see Fig. 4) and we expect to see a pitchfork bifurcation emergence, i.e. emergence of additional two stable equilibria and loss of stability of the current equilibrium. In Fig. 8 we saw that original equilibrium point B was attracting therefore stable. As we decrease σ to 0.191009 (still unrealistic value) B is still stable attractor. However, as we continue to decrease σ further, two new equilibria (A and C) emerge while B looses stability[6].

Fig. 8: Emergence of the Pitchfork Bifurcation

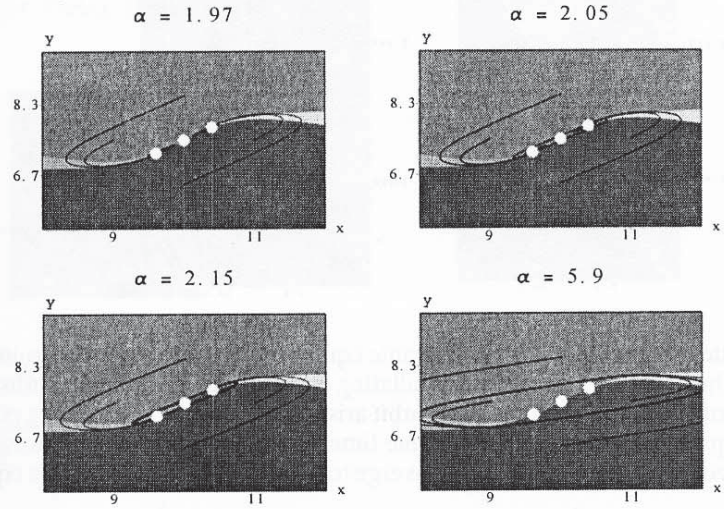


6 Note, we have used a parameter arrangement such that time path trajectories converge in an oscillating manner, i.e. α is rather large.

3.3. Two equilibria, Hopf and Homoclinic bifurcations

The Hopf and Homoclinic bifurcation can be surely regarded as the most interesting behavioral patterns in this type of model. To observe them we will again work with „sub critical" value of propensity to save (σ = 0.15). This would ensure existence of three equilibria and we will focus on changes in . Increasing the speed of adjustment the private sector's reaction to changes in demand becomes very fast. Rapid reaction usually lead to overshooting and therefore to oscillating convergence paths. This time, however, we have two stable equilibria. We first set α = 1.97 (top left Fig. 9). As expected the paths converge in oscillating manner to outer equilibria A and C. Next we increase α to 2.05. The frequency of oscillations increases. Trajectories, however still converge to the stable points A and C. On the following picture (α = 2.15) the trajectories oscillate even longer before they reach their attracting points. And finally the last picture reveals completion of Hopf bifurcation. Trajectories no longer converge to equilibrium points but to an orbit developed around the three equilibria. Note, that no matter whether we choose initial points from inside or outside the orbit system always converges to it. We have also generated a figure displaying basins of attraction for the first two cases α = 1.97 and α = 2.05. It shows which initial points converge to point A or C.

Fig. 9: Bistability with Hopf Bifurcation and Homoclinic Orbit



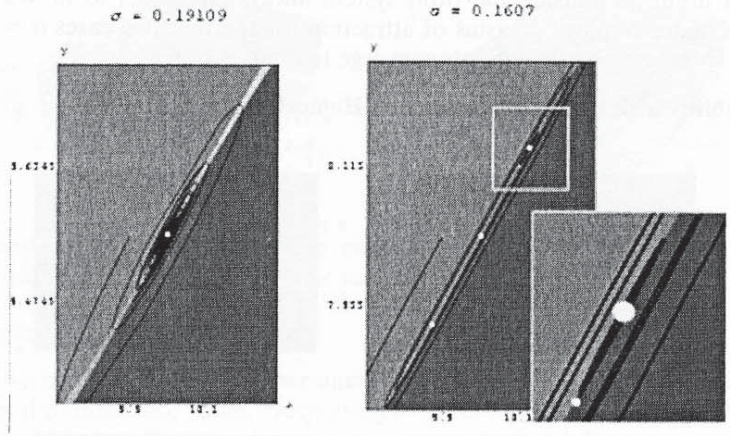
If we pick an initial value from the light area the system will converge to the left equilibrium point A. If we start from a point in the dark area the system will converge to the right equilibrium point C. The basins of attraction preserve very similar shape for the first two pictures of Fig. 10 therefore we present only one figure for both cases[7].

7 Regarding the last two picture (final stages of Hopf bifurcation) it does not make sense to generate basins of attraction for these cases as all points in the {Y,K} space eventually converge to the orbit therefore there is no distinction between the initial values.

Fig. 10: Basins of attraction before Hopf bifurcation.

Finally we study emergence of Homoclinic orbit. Under our particular set up and parameter arrangement we have located Homoclinic orbit by setting $\alpha = 1.71$ and changing σ (as usual values of σ are given above plots). Other parameter values remain unchanged. The Fig. 11 uncovers the system behavior.

Fig. 11: Emergence of Homoclinic orbit



Our system thus evolves from one stable equilibrium into three equilibrium situation. Speed of adjustment of 1.71 ensures oscillating convergence of the time paths. When we reach value of $\sigma = 0.1607$ a Homoclinic orbit arises. If we choose as a starting point a point outside the path the system will after some time converge to the orbit. However, starting points located within the orbit do not converge to it but converge to the stable equilibria.

3. Summary

In this paper we have analyzed dynamic behavior of a modified Kaldor model. In the usual set up dynamics of Kaldor model is analyzed with investment function given as a simple S function such as Arcus or similar. We have decided to use different functional form – the logistic function. In the investment function we have implemented parameter for an interest rate. In our future work we intend to study impact of interest rate changes on the Kaldor model. Our main task, however was to proceed with a bifurcation analysis. Consequently we have located and described Hopf bifurcation in one equilibria framework, Pitchfork bifurcation, Hopf in three equilibria framework and the Homoclinic bifurcation. We also have located critical parameter values and regions in parameter space.

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Lokální stabilita a bifurkace v Kaldorově modelu

Roman Binter – Lukáš Vácha

Abstrakt

V tomto článku analzyujeme diskrétní verzi Kaldorova modelu. Typickým rysem modelu je S tvar investiční funkce, který zajišťuje existenci jednoho nebo tří rovnovážných stavů modelu. Z důvodů přehlednosti uvažujeme lineární formu úsporové funkce. Toto zjednodušení nemá vliv na interpretaci výsledků. Naším cílem je studium změn dynamického chování modelu v důsledku změny parametrů. Soustředíme se na přechod modelu ze stavu jednoho equilibria do stavu tří equilibrií a s tím spojené bifurkační chování.

Klíčová slova: ekonomická dynamika; bifurkace; Kaldorův model.

Local Stability and Bifurcations in Kaldor Model

Abstract

We analyze a discrete version of a simple Kaldor model. As is typical for Kaldor model we consider an S shaped investment function. This leads to either a one or three equilibria of the model. For simplicity reasons we do not consider an S shape saving function as assumed in the original Kaldor paper. This does not affect any analytical conclusions as for presentation of dynamic properties nonlinear investment function is sufficient. Our aim is to study changes in the model dynamics under varying parameters. We study transition between one and two equilibria setup and also each of the set up separately.

Key words: economic dynamics; bifurcations; Kaldor model.

JEL classification: C62, E32, E12.